

# POINTWISE PARTIAL HYPERBOLICITY IN 3-DIMENSIONAL NILMANIFOLDS

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**ABSTRACT.** We show the existence of a family of manifolds on which all (pointwise or absolutely) partially hyperbolic systems are dynamically coherent. This family is the set of 3-manifolds with nilpotent, non-abelian fundamental group. We further classify the partially hyperbolic systems on these manifolds up to leaf conjugacy. We also classify those systems on the 3-torus which do not have an attracting or repelling periodic 2-torus. These classification results allow us to prove some dynamical consequences, including existence and uniqueness results for measures of maximal entropy and quasi-attractors.

**Keywords:** Partial hyperbolicity (pointwise), Dynamical Coherence, Global Product Structure, Leaf conjugacy.

**MSC 2000:** 37C05, 37C20, 37C25, 37C29, 37D30, 57R30.

## 1. INTRODUCTION

Since the beginning of its study, there have been two competing candidates for the definition of partial hyperbolicity. In both definitions, there is a continuous splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

invariant under the derivative  $Df$  of the diffeomorphism  $f : M \rightarrow M$  and where the strong expansion of the unstable bundle  $E^u$  and the strong contraction of the stable bundle  $E^s$  dominate any expansion or contraction on the center  $E^c$ . The distinction between the two definitions is in the exact nature of the domination. If (for some Riemannian metric on  $M$ ) the inequalities

$$\|Dfv^s\| < \|Dfv^c\| < \|Dfv^u\| \quad \text{and} \quad \|Dfv^s\| < 1 < \|Dfv^u\|$$

are satisfied for each point  $x \in M$  and for unit vectors  $v^s \in E^s(x)$ ,  $v^c \in E^c(x)$ , and  $v^u \in E^u(x)$ , then  $f$  is *pointwise* (or *relatively*) partially hyperbolic. If  $f$  also satisfies the condition that there are global constants  $\lambda, \hat{\gamma}, \gamma, \mu$  independent of  $x$  such that

$$\|Dfv^s\| < \lambda < \hat{\gamma} < \|Dfv^c\| < \gamma < \mu < \|Dfv^u\|$$

then  $f$  is *absolutely* partially hyperbolic. See [HPS] where both notions are discussed.

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Only recently have important differences between the two notions come to light.

In every (pointwise or absolutely) partially hyperbolic system, there are unique foliations  $\mathcal{W}^u$  and  $\mathcal{W}^s$  tangent to  $E^u$  and  $E^s$  ([HPS]), but there is not always a foliation tangent to  $E^c$  (see [BuW<sub>1</sub>]). A partially hyperbolic diffeomorphism  $f$  is *dynamically coherent* if there are  $f$ -invariant foliations tangent to  $E^{cs} = E^s \oplus E^c$ ,  $E^{cu} = E^c \oplus E^u$ , (and consequently also to  $E^c$ ).

Brin, Burago, and Ivanov proved that every absolutely partially hyperbolic system on the 3-torus is dynamically coherent [BBI<sub>2</sub>]. Inspired by this result, Hertz, Hertz, and Ures attempted to extend it to all pointwise partially hyperbolic systems, but this line of research led them to discover a counterexample. There are pointwise partially hyperbolic diffeomorphisms on  $\mathbb{T}^3$  for which there is no foliation tangent to the center direction [RHRHU<sub>4</sub>]. Hertz, Hertz, and Ures further asked if there are any manifolds on which all pointwise partially hyperbolic systems are dynamically coherent. In this paper, we answer this question in the affirmative.

**Theorem 1.1.** *Suppose  $M$  is a 3-manifold with (virtually) nilpotent fundamental group. If  $M$  is not finitely covered by  $\mathbb{T}^3$ , then every pointwise partially hyperbolic diffeomorphism on  $M$  is dynamically coherent.*

This builds on results, obtained independently in [H<sub>2</sub>] and [Par], in the absolute partially hyperbolic setting.

Only certain 3-manifolds can support partially hyperbolic diffeomorphisms. In particular, if  $\pi_1(M)$  is (virtually) nilpotent and  $M$  supports partially hyperbolic diffeomorphisms, then  $M$  is (finitely covered by) a circle bundle over a 2-torus [Par]. Therefore, Theorem 1.1 will follow as a consequence of:

**Theorem 1.2.** *If  $M$  is a non-trivial circle bundle over  $\mathbb{T}^2$  and  $f : M \rightarrow M$  is pointwise partially hyperbolic, then  $f$  is dynamically coherent and there is a unique  $f$ -invariant foliation tangent to each of the bundles. Further, the bundle projection  $\pi : M \rightarrow \mathbb{T}^2$  may be chosen so that every leaf of the center foliation is a fiber of the bundle.*

Beyond dynamical coherence, we also consider the classification problem for pointwise partially hyperbolic systems. In studying Anosov flows, the natural notion of equivalence is topological equivalence. Two flows are topologically equivalent if there is a homeomorphism which maps orbits of one flow to orbits of the other and which preserves the orientations of the orbits.

The natural notion of equivalence for partially hyperbolic systems is leaf conjugacy, as introduced in [HPS]. Two dynamically coherent partially hyperbolic diffeomorphisms  $f$  and  $g$  are *leaf conjugate* if there is a homeomorphism  $h$  which maps each center leaf  $L$  of  $f$  to a center leaf of  $g$  and  $hf(L) = gh(L)$ .

For every diffeomorphism  $f$  of the 3-torus, there is a unique linear map  $A_f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $A_f(\mathbb{Z}^3) = \mathbb{Z}^3$  and, viewing  $\mathbb{T}^3$  as the quotient  $\mathbb{R}^3/\mathbb{Z}^3$ , the quotiented map  $A_f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is homotopic to  $f$ . We call  $A_f$  the *linear part* of  $f$ . If  $f$  is absolutely partially hyperbolic, then  $A_f$  has a partially hyperbolic splitting and the diffeomorphisms  $f$  and  $A_f$  are leaf conjugate [H].

If  $f$  is pointwise partially hyperbolic, it may not have a center foliation ([RHRHU<sub>4</sub>]), in which case, it is not possible to define a leaf conjugacy. However, in [Pot<sub>1</sub>] it was shown that there is only one obstruction to dynamical coherence in  $\mathbb{T}^3$  and here we show that this is also the only obstruction to extending the classification.

**Theorem 1.3.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be pointwise partially hyperbolic. Then, either*

- *$f$  is not dynamically coherent and there is a periodic 2-torus  $T = f^k(T)$  tangent either to  $E^c \oplus E^u$  or  $E^c \oplus E^s$ , or*
- *$f$  is dynamically coherent and leaf conjugate to its linear part.*

In the first case above, such a torus  $T$  is transverse either to  $E^u$  or  $E^s$  and is therefore either an attractor or a repeller. Such phenomena are impossible when the chain-recurrence set  $\mathcal{R}(f)$  is all of  $\mathbb{T}^3$  (see [BDV, Chapter 10]).

**Corollary 1.4.** *If  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is pointwise partially hyperbolic and  $\mathcal{R}(f) = \mathbb{T}^3$ , then  $f$  is dynamically coherent and leaf conjugate to its linear part.*

Note that if the non-wandering set  $\Omega(f)$  is all of  $\mathbb{T}^3$  (for example when  $f$  is transitive or when  $f$  is volume-preserving) the previous corollary applies.

One can also show that a torus  $T$  as in Theorem 1.3 cannot exist when the linear part is hyperbolic (see for example Proposition A.1 of [Pot<sub>1</sub>]).

**Corollary 1.5.** *If  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is pointwise partially hyperbolic and its linear part  $A_f$  has no eigenvalues of modulus one, then  $f$  is dynamically coherent and leaf conjugate to  $A_f$ .*

The proof of Theorem 1.3 combines results from [H], where leaf conjugacy was obtained for absolutely partially hyperbolic systems, and [Pot<sub>1</sub>], where dynamical coherence was studied in the pointwise case. This proof is given in section 3 after some preliminaries are introduced in section 2.

For the manifolds considered in Theorem 1.2, all of the systems are dynamically coherent and we can classify all of them. If  $M$  is a circle bundle over  $\mathbb{T}^2$ , it is a *nilmanifold*, i.e. a compact quotient  $G/\Gamma$  of a nilpotent Lie group  $G$ . For a diffeomorphism  $f : M \rightarrow M$ , there is a unique Lie group automorphism on  $G$  which descends to a map  $\Phi_f : M \rightarrow M$  homotopic to  $f$ . Call  $\Phi_f$  the *algebraic part* of  $f$ .

**Theorem 1.6.** *Suppose  $M$  is a 3-dimensional nilmanifold,  $M \neq \mathbb{T}^3$ , and  $f : M \rightarrow M$  is pointwise partially hyperbolic. Then,  $f$  is leaf conjugate to its algebraic part  $\Phi_f$ .*

In 2001, Pujals made a conjecture on transitive partially hyperbolic diffeomorphisms which we paraphrase here (see [BoW]).

**Conjecture 1.7** (Pujals (2001)). *Up to a finite cover, every transitive partially hyperbolic diffeomorphism on a 3-manifold is leaf conjugate to*

- *an Anosov diffeomorphism on  $\mathbb{T}^3$ ,*
- *a time-one map of an Anosov flow, or*
- *a topological skew product over an Anosov map on  $\mathbb{T}^2$ .*

More recently (in 2009), Hertz, Hertz, and Ures conjectured that transitivity could be replaced by dynamical coherence in the previous conjecture. Moreover, they posed the following conjecture which, if proven to be true implies in particular that transitive partially hyperbolic diffeomorphisms are dynamically coherent.

**Conjecture 1.8** (Hertz, Hertz, Ures (2009)). *If a partially hyperbolic diffeomorphism  $f$  on a 3-manifold is not dynamically coherent, there is a periodic torus  $T = f^k(T)$  tangent either to  $E^c \oplus E^u$  or  $E^c \oplus E^s$ .*

Both conjectures were posed in the pointwise case, and the results of this paper show that both conjectures are true when the manifold in question has (virtually) nilpotent fundamental group. As we mentioned, this last conjecture in the case of  $\mathbb{T}^3$  was already established in [Pot<sub>1</sub>]. We mention that in a forthcoming paper ([HP]) we plan to extend our results to all 3-manifolds with *solvable* fundamental group.

One obvious reason to study pointwise partially hyperbolic systems over absolutely partially hyperbolic systems is that it is more general. One family of diffeomorphisms properly includes the other. Another important motivation is the study of robust transitivity and stable ergodicity (see [BDV, Wi]): Díaz, Pujals, and Ures proved that every  $C^1$  robustly transitive diffeomorphism of a 3-manifold is partially hyperbolic in the weak sense, that is, it satisfies the definition given at the start of this paper, with the possible caveat that one of the bundles  $E^s$ ,  $E^c$ , or  $E^u$  may be zero [DPU]. Their theorem is definitely a pointwise theorem, as there are robustly transitive diffeomorphisms which are not absolutely partially hyperbolic. The result shows that pointwise partial hyperbolicity is a notion which arises naturally when studying the space of  $C^1$  diffeomorphisms.

We remark that once the topological classification is obtained, further dynamical consequences follow. Section 6 explores some more-or-less direct consequences of our result. First, we study the existence and finiteness of maximal entropy measures which are a direct application of our main results and previous ones [RHRHTU, U]. Then, we obtain

another dynamical consequence which holds for partially hyperbolic diffeomorphisms of non-toral nilmanifolds:

**Proposition 1.9.** *Let  $f : M \rightarrow M$  a partially hyperbolic diffeomorphism of a nilmanifold  $M \neq \mathbb{T}^3$ . Then, the foliations  $\mathcal{W}^u$  and  $\mathcal{W}^s$  have a unique minimal set (in particular, a unique quasi-attractor<sup>1</sup>).*

**Notation.** Throughout this paper “partially hyperbolic” without further qualifiers is taken to mean pointwise partially hyperbolic and where all three bundles  $E^u$ ,  $E^c$ , and  $E^s$  are non-zero.

## 2. BRANCHING FOLIATIONS

In this section, we introduce the notions of “almost aligned” and “asymptotic” for branching and non-branching foliations, and review the results of Burago and Ivanov.

A (*complete*) *surface* in a 3-manifold  $M$  is a  $C^1$  immersion  $\iota : U \rightarrow M$  of a connected smooth 2-dimensional manifold without boundary  $U$  which is complete with the metric induced by the metric on  $M$  pulled back by  $\iota$ .

A *branching foliation* on  $M$  is a collection of complete surfaces tangent to a given continuous 2-dimensional distribution on  $M$  such that:

- Every point is in the image of at least one surface.
- There are no topological crossings between any two surfaces of the collection. That is, no curve lying on one leaf can cross through another leaf.
- It is complete in the following sense: if  $x_k \rightarrow x$  and  $\iota_k$  are surfaces of the partition having  $x_k$  in its image we have that  $\iota_k$  converges in the  $C^1$ -topology to a surface of the collection with  $x$  in its image (see [BI] Lemma 7.1).

The image  $\iota(U)$  of each surface in a branching foliation is called a *leaf*.

**Theorem 2.1** (Burago-Ivanov [BI], Theorem 4.1). *If  $f$  is a partially hyperbolic diffeomorphism on a 3-manifold  $M$ , such that the bundles  $E^s$ ,  $E^c$  and  $E^u$  are orientable and the orientation is preserved by  $Df$ , then, there is a (not necessarily unique)  $f$ -invariant branching foliation  $\mathcal{F}_{bran}^{cs}$  tangent to  $E^{cs}$ . Further, any curve tangent to  $E^s$  lies in a single leaf of  $\mathcal{F}_{bran}^{cs}$ .*

A similar foliation  $\mathcal{F}_{bran}^{cu}$  is defined tangent to  $E^{cu}$  under the same hypotheses.

**Remark.** In proving results of dynamical coherence, we will need to show in some cases that the branching foliation given by this theorem is a true foliation. A helpful observation is that if  $\mathcal{F}$  is a branching foliation such that every point in  $M$  belongs to a unique leaf, then  $\mathcal{F}$  is indeed a true foliation (see Proposition 1.6 and Remark 1.10 in

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<sup>1</sup>See section 6 for a definition.

[BoW]). A *foliation* for us will mean a  $C^{1,0+}$  foliation in the notation of [CC], that is, a  $C^0$ -foliation with  $C^1$ -leaves tangent to a continuous distribution.

A surface  $\iota : S \rightarrow M$  can be lifted to a surface  $\tilde{\iota} : \tilde{S} \rightarrow \tilde{M}$  where  $\tilde{S}$  and  $\tilde{M}$  are the universal covers. This lift is not unique in general. For a branching foliation  $\mathcal{F}$  on  $M$ , consider the collection of all possible lifts of all surfaces. This collection defines a unique branching foliation  $\tilde{\mathcal{F}}$  called the lift of  $\mathcal{F}$  to  $\tilde{M}$ . A branching foliation  $\mathcal{F}_1$  is *almost aligned* with a branching foliation  $\mathcal{F}_2$  if there is  $R > 0$  such that each leaf of  $\tilde{\mathcal{F}}_1$  lies in the  $R$ -neighborhood of a leaf of  $\tilde{\mathcal{F}}_2$ . Note that this defines a relation on the set of foliations which is transitive, but not necessarily symmetric.

Burago and Ivanov further proved that from any branching foliation, a non-branching foliation can be constructed by deforming each leaf by an arbitrarily small amount [BI] (Section 7). In the partially hyperbolic setting, if the branching foliation is transverse to the unstable foliation, the new foliation is also transverse, and by this virtue, it does not contain a Reeb component [BI] (Section 2). These results imply the following.

**Theorem 2.2** (Burago-Ivanov [BI]). *Under the hypotheses of Theorem 2.1, there is a (non-branching)  $C^{1,0+}$  Reebless foliation  $\mathcal{W}$  such that  $\mathcal{W}$  is almost aligned with  $\mathcal{F}_{bran}^{cs}$  and  $\mathcal{F}_{bran}^{cs}$  is almost aligned with  $\mathcal{W}$ . A similar foliation exists for  $\mathcal{F}_{bran}^{cu}$ .*

Note that the new foliation  $\mathcal{W}$  is neither  $f$ -invariant nor tangent to  $E^{cs}$ .

In the specific case of the torus  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ , we define a notion of the “asymptotic” behavior of a foliation. A linear subspace  $V \subset \mathbb{R}^d$  defines a linear foliation  $\tilde{\mathcal{F}}_V$  where the leaves are the fibers of the orthogonal projection  $\pi : \mathbb{R}^d \rightarrow V^\perp$ . A branching foliation  $\tilde{\mathcal{F}}$  is *asymptotic* to  $\tilde{\mathcal{F}}_V$  if for  $\varepsilon > 0$  there is  $K > 0$  such that

$$\|x - y\| > K \quad \Rightarrow \quad \|\pi x - \pi y\| < \varepsilon \|x - y\|$$

for all  $x, y$  on the same leaf of  $\tilde{\mathcal{F}}$ . Note that if  $\tilde{\mathcal{F}}$  is almost aligned with  $\tilde{\mathcal{F}}_V$  then  $\tilde{\mathcal{F}}$  is asymptotic to  $\tilde{\mathcal{F}}_V$ .

As a final definition, a foliation  $\tilde{\mathcal{W}}$  on a manifold  $\tilde{M}$  is *quasi-isometric* if there is a global constant  $Q > 1$  such that

$$d_{\tilde{\mathcal{W}}}(x, y) < Q d_{\tilde{M}}(x, y) + Q$$

for any points  $x$  and  $y$  on the same leaf of  $\tilde{\mathcal{W}}$ . Here,  $d_{\tilde{\mathcal{W}}}$  denotes the distance inside a leaf of the foliation.

Figure 1 gives a graphical illustration of these definitions.

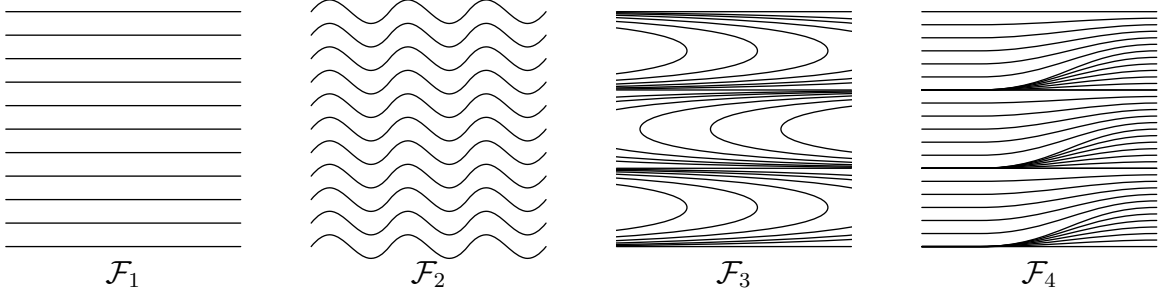


FIGURE 1. A graphical depiction of four examples of branching foliations on the plane. Only  $\mathcal{F}_4$  exhibits branching; the others are true foliations. For any  $1 \leq i, j \leq 4$ , one can verify that  $\mathcal{F}_i$  is almost aligned with  $\mathcal{F}_j$ . The foliation  $\mathcal{F}_1$  is linear and the other foliations are asymptotic to  $\mathcal{F}_1$ . Finally,  $\mathcal{F}_3$  is the only one which is not quasi-isometric.

### 3. LEAF CONJUGACY ON THE 3-TORUS

The main result of [H] is the following.

**Theorem 3.1.** *Let  $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$  be an absolutely partially hyperbolic diffeomorphism. Suppose*

- $\tilde{W}_f^\alpha$  is a quasi-isometric foliation ( $\alpha = u, s$ ), and
- $E_f^c$  is one-dimensional.

*Then,  $f$  is leaf conjugate to its linear part.*

While the theorem is stated under the stronger hypothesis of absolute partial hyperbolicity, many of the arguments of the proof apply equally well in the pointwise case. In fact, absolute partial hyperbolicity is used only in the beginning of one of the sections (Chapter 2, titled “Nice Properties of the Invariant Manifolds”).

In this paper’s introduction, we used  $A_f$  to denote the linear part of  $f$ , whereas in [H] it is called the “linearization” and denoted by  $g$ . In this section, we use the symbols  $A_f$  and  $g$  interchangeably depending on the context.

At the start of the proof in [H], the constants  $0 < \lambda < \hat{\gamma} < \gamma < \mu$  associated to the absolutely partially hyperbolic splitting of  $f$  are used to define a splitting for  $g$ . Next, the quasi-isometry assumption is used to show dynamical coherence. This follows from a result of Brin which holds only in the absolutely partially hyperbolic case [Br]. Then, [H] states and proves a number of propositions comparing the foliations of  $f$  and  $g$ . The proofs of exactly three of these propositions (numbered (2.3), (2.5), and (2.7)) rely on absolute instead of pointwise partial hyperbolicity. Using the definitions given in this paper, these three propositions may be stated as follows.

- I.  $\tilde{W}_f^\alpha$  is asymptotic to  $E_g^\alpha$  ( $\alpha = u, s$ ).

II.  $\tilde{\mathcal{W}}_f^\alpha$  is almost aligned with  $\tilde{\mathcal{W}}_g^\alpha$  ( $\alpha = cu, cs$ ).

III. For all  $x \in \mathbb{R}^d$ ,  $\tilde{\mathcal{W}}_f^{cs}(x) \cap \tilde{\mathcal{W}}_f^u(x) = \{x\}$  and  $\tilde{\mathcal{W}}_f^{cu}(x) \cap \tilde{\mathcal{W}}_f^s(x) = \{x\}$ .

The version of II in [H] also states that  $\tilde{\mathcal{W}}_f^c$  is almost aligned with  $\tilde{\mathcal{W}}_g^c$ , but this easily follows as a consequence of II as given above. While the proof of III in [H] relies on absolute partial hyperbolicity, it can be replaced by a short proof that works in the pointwise case.

*Proof of III assuming I and II.* Suppose  $x \neq y \in \tilde{\mathcal{W}}_f^{cs}(x) \cap \tilde{\mathcal{W}}_f^u(x)$  and define  $v_n = f^n(x) - f^n(y)$ . Quasi-isometry of the unstable foliation implies that  $\|v_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ . Then, property I implies that the angle of  $v_n$  with  $E_g^u$  goes to zero and property II implies that the angle with  $E_g^{cs}$  goes to zero. These cannot both be true.  $\square$

As the rest of the proof of Theorem 3.1 follows assuming only pointwise partial hyperbolicity, we may reformulate the statement thus.

**Theorem 3.2.** *Let  $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$  be a pointwise partially hyperbolic diffeomorphism with linear part  $g : \mathbb{T}^d \rightarrow \mathbb{T}^d$ . Suppose  $g$  has a (linear) partially hyperbolic splitting and that*

- $f$  is dynamically coherent,
- $\tilde{\mathcal{W}}_f^\alpha$  is a quasi-isometric foliation ( $\alpha = u, s$ ),
- $E_f^c$  is one-dimensional, and
- $f$  and  $g$  satisfy properties I and II above.

*Then,  $f$  is leaf conjugate to  $g$ .*

**Remark.** The constants  $\lambda$  and  $\mu$  from the absolutely partially hyperbolic splitting are used throughout the exposition in [H]. However, apart from the special cases mentioned above, all we need are constants  $C_{\text{ph}} > 1$  and  $0 < \lambda < 1 < \mu$  satisfying

$$\begin{aligned} \frac{1}{C_{\text{ph}}} \mu^n \|v\| &< \|Df^n v\| && \text{for } v \in E_f^u(x) \setminus \{0\}, \\ \|Df^n v\| &< C_{\text{ph}} \lambda^n \|v\| && \text{for } v \in E_f^s(x) \setminus \{0\}. \end{aligned}$$

Such constants exist for any pointwise partially hyperbolic system.

This finishes the discussion for diffeomorphisms on a torus  $\mathbb{T}^d$  of general dimension  $d \geq 3$ . In the specific case of dimension three, we combine Theorem 3.2 with the results in [Pot<sub>1</sub>] to prove Theorem 1.3. First, we note that the linear part  $g = A_f$  is partially hyperbolic.

**Proposition 3.3.** *If  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is partially hyperbolic, the linear part  $A_f$  has real eigenvalues  $\lambda_1, \lambda_2$  and  $\lambda_3$  where*

$$|\lambda_1| < |\lambda_2| < |\lambda_3| \quad \text{and} \quad |\lambda_1| < 1 < |\lambda_3|.$$

*The associated eigenspaces define a partially hyperbolic splitting for  $A_f$ .*



*Proof.* Let  $\lambda_1, \lambda_2$  and  $\lambda_3$  be the (possibly complex) eigenvalues of  $A_f$ , ordered so that  $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3|$ . Burago and Ivanov proved that  $|\lambda_1| < 1 < |\lambda_3|$ ; see [BI] (Theorem 1.2). If  $|\lambda_2| = 1$ , we are done, since non-real eigenvalues must come in conjugate pairs. If  $|\lambda_2| > 1$ , then [Pot<sub>1</sub>] (Theorem A) shows that  $\lambda_2$  and  $\lambda_3$  are real and distinct, and  $\lambda_1$ , as the only eigenvalue with modulus smaller than 1, must be real as well. Finally, if  $|\lambda_2| < 1$ , consider  $f^{-1}$  instead.  $\square$

To show that the unstable foliation of  $f$  is asymptotic to the unstable foliation of  $A_f$ , we need a technical lemma.

**Lemma 3.4.** *Suppose*

- $\mathcal{F}$  is a one-dimensional foliation on  $\mathbb{T}^3$ ,
- $\mathcal{F}$  is invariant under a diffeomorphism  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ ,
- the linear part  $A_f$  of  $f$  is partially hyperbolic and therefore defines a plane  $P$  through the origin tangent to  $E_{A_f}^{cs}$ , and
- there is  $K > 0$  and a cone  $\mathcal{E}$  transverse to  $P$  such that

$$\|y - x\| > K \quad \Rightarrow \quad y - x \in \mathcal{E}$$

for all  $x \in \mathbb{R}^3$  and  $y \in \tilde{\mathcal{F}}(x)$ .

Then,  $\tilde{\mathcal{F}}$  is asymptotic to the unstable foliation of  $A_f$ .

Here, we take a cone transverse to  $P$  to mean a set of the form

$$\mathcal{E} = \{v \in \mathbb{R}^3 : \|\pi v\| < c\|v\|\}$$

where  $0 < c < 1$  and where  $\pi : \mathbb{R}^3 \rightarrow P$  is any projection. The proof of Lemma 3.4 relies on two properties. First, that as  $k \rightarrow \infty$ , the angle between a vector  $v \in A_f^k(\mathcal{E})$  and the one-dimensional subspace  $E_{A_f}^u$  tends uniformly to zero. Second, that for fixed  $k$ , as  $\|y - x\|$  goes to infinity, the angle between  $f^k(y) - f^k(x)$  and  $A_f^k(y) - A_f^k(x)$  tends uniformly to zero. We leave the details to the reader.

Several of the results of [Pot<sub>1</sub>] are used to establish the leaf conjugacy. We compile them all into the following statement.

**Theorem 3.5.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a partially hyperbolic diffeomorphism such that there is no periodic 2-torus tangent to  $E^s \oplus E^c$ . Then:*

- (i) *There is a unique  $f$ -invariant foliation  $\mathcal{W}_f^{cs}$  tangent to  $E_f^{cs}$ .*
- (ii)  *$\tilde{\mathcal{W}}_f^{cs}$  is almost aligned with  $\tilde{\mathcal{W}}_{A_f}^{cs}$ .*
- (iii) *Each leaf of  $\tilde{\mathcal{W}}_f^{cs}$  intersects each leaf of  $\tilde{\mathcal{W}}_f^u$  exactly once.  
(That is, the two foliations have global product structure.)*
- (iv)  *$\tilde{\mathcal{W}}_f^u$  is a quasi-isometric foliation.*
- (v)  *$\tilde{\mathcal{W}}_f^u$  is asymptotic to  $\tilde{\mathcal{W}}_{A_f}^u$ .*

*Proof.* The first item is a restatement of [Pot<sub>1</sub>] (Theorem B). The second and third items follow from [Pot<sub>1</sub>] (Proposition A.1) in the case where  $A_f$  is Anosov, or from [Pot<sub>1</sub>] (Proposition 8.7) in the case where it is not. With these established, [Pot<sub>1</sub>] (Proposition 6.9) implies that  $\tilde{\mathcal{W}}_f^u$  is quasi-isometric (iv) and that there is a cone  $\mathcal{E}$  as in Lemma 3.4 above. The last item then follows from this lemma.  $\square$

Note that this result also holds with the roles of the stable and unstable bundles exchanged. Theorem 1.3 now follows easily as a combination of Theorem 3.2, Proposition 3.3, and Theorem 3.5.

#### 4. NUMBERING

The next section extensively references results in section 4 of [H<sub>2</sub>]. To avoid any possible confusion between that paper and this one, we do not state any results in this paper's section 4.

#### 5. NON-TORAL NILMANIFOLDS

There are several ways to view nilmanifolds in dimension three. One is as quotients of nilpotent Lie groups. Another is as bundles of the 2-torus over the circle. While in this section, we only consider the former, we will use the latter view in Appendix B.

Let  $\mathcal{H}$  denote the Heisenberg group, the group of all real-valued matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Fix a co-compact subgroup  $\Gamma$ . Then  $\mathcal{H}/\Gamma$  is a compact manifold and every non-toral three-dimensional nilmanifold is of this form. For a more detailed introduction, see [H<sub>2</sub>]. Observe that if  $\pi : \mathcal{H} \rightarrow \mathbb{R}$  is a Lie group homomorphism, it must be of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto ax + by$$

for constants  $a, b \in \mathbb{R}$ . If  $\pi$  is non-zero, its level sets define a codimension one foliation  $\tilde{\mathcal{F}}_\pi$  on  $\mathcal{H}$  which quotients down to a foliation  $\mathcal{F}_\pi$  on the nilmanifold  $\mathcal{H}/\Gamma$ . Plante showed that any Reebless  $C^2$  foliation of  $\mathcal{H}/\Gamma$  must be almost aligned to  $\mathcal{F}_\pi$  for some  $\pi$  [P]. We need a similar result for foliations which are  $C^{1,0+}$ , that is,  $C^0$  with  $C^1$  leaves tangent to a  $C^0$  distribution. We prove the following.

**Proposition 5.1.** *Every Reebless  $C^{1,0+}$  foliation on  $\mathcal{H}/\Gamma$  is almost aligned with some  $\mathcal{F}_\pi$ .*

There are two approaches to proving this. One is to adapt the proof of Plante to the  $C^{1,0+}$  setting. The other is to extend the techniques used by Brin, Burago, and Ivanov on the 3-torus to other manifolds. Both approaches work, and since both techniques may be useful in the future if applied to other manifolds, we give both proofs in the appendices. Appendix A gives a self contained proof of this result using some algebraic arguments à la Brin-Burago-Ivanov. Appendix B provides a more geometric proof which uses the classification of such foliations in  $\mathbb{T}^3$  and some general position results ([Rou, Ga]) and gives a classification for every torus bundle over the circle (this will be used in [HP]). In this section, we assume Proposition 5.1 and show how it can be used to prove Theorems 1.2 and 1.6.

**Remark.** To keep the presentation short, we only reprove those parts of [H<sub>2</sub>] which rely on absolute partial hyperbolicity. This has the unfortunate consequence that the proof of dynamical coherence for the pointwise case is split between different papers and is not presented in one place from start to finish. As such, we present here a rough outline of the proof as a whole.

If  $J$  is a small unstable curve on the universal cover  $\mathcal{H}$ , then the length of  $\tilde{f}^k(J)$  grows exponentially fast. By the results of Brin, Burago, and Ivanov,  $U_1(\tilde{f}^k(J))$  also grows exponentially fast in volume. Here,  $U_1(X)$  is all points at distance at most 1 from a point in  $X$ . Comparing  $\tilde{f}$  with its algebraic part  $\Phi = \Phi_f : \mathcal{H} \rightarrow \mathcal{H}$ , one can prove that such exponential expansion is only possible if  $\tilde{f}^k(J)$  lies more-or-less in the unstable direction of  $\Phi$ . Since these unstable curves lie in leaves of the branching foliation  $\tilde{\mathcal{F}}_{bran}^{cu}$ , it follows from Proposition 5.1 that  $\tilde{\mathcal{F}}_{bran}^{cu}$  is almost aligned with the center-unstable foliation of  $\Phi$  as no other foliations of the form  $\tilde{\mathcal{F}}_\pi$  have long curves in the same direction.

Similarly,  $\tilde{\mathcal{F}}_{bran}^{cs}$  is almost aligned with the center-stable foliation of  $\Phi$  and there is  $R > 0$  such that if  $p$  and  $q$  lie on the same leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  then  $q$  is at distance at most  $R$  from  $\tilde{W}_\Phi^{cs}(p)$ . If  $\mathcal{F}_{bran}^{cs}$  is genuinely branching and not a true foliation, there are two distinct leaves of  $\tilde{\mathcal{F}}_{bran}^{cs}$  which intersect in a common point  $p$  and one can find a small unstable arc  $J$  which lies in the space between these two leaves. For all  $k > 0$ ,  $\tilde{f}^k(J)$  lies between two leaves of  $\tilde{\mathcal{F}}_{bran}^{cs}$  which pass through the point  $\tilde{f}^k(p)$  and so all points in  $f^k(J)$  are at distance at most  $R$  from  $\tilde{W}_\Phi^{cs}(f^k(p))$ . This contradicts the above-stated fact that  $f^k(J)$  must grow exponentially fast in the unstable direction of  $\Phi$ . Thus,  $\mathcal{F}_{bran}^{cs}$  is a non-branching foliation, and  $f$  is dynamically coherent. Again, we stress that this is a non-rigorous outline, and the rigorous proof follows below.

**5.1. The orientable case.** Suppose  $f : \mathcal{H}/\Gamma \rightarrow \mathcal{H}/\Gamma$  is partially hyperbolic. To keep things simple, we make the unjustified assumption that the subbundles  $E^u$ ,  $E^c$ ,  $E^s$  are oriented, and that  $Df$  preserves these orientations. The next subsection shows how the general case can be deduced from this special case. Under these assumptions, Theorem

2.1 applies and there are  $f$ -invariant branching foliations  $\mathcal{F}_{bran}^{cu}$  and  $\mathcal{F}_{bran}^{cs}$  tangent to  $E^{cu}$  and  $E^{cs}$ .

**Lemma 5.2.**  $\mathcal{F}_{bran}^{cu}$  is almost aligned with some  $\mathcal{F}_\pi$ .

*Proof.* This follows immediately from Proposition 5.1 and Theorem 2.2.  $\square$

If  $\pi : \mathcal{H} \rightarrow \mathbb{R}$  is a Lie group homomorphism, there is  $C > 0$  such that

$$|\pi(p) - \pi(q)| < Cd(p, q)$$

for all  $p, q \in \mathcal{H}$ . This can be shown by adapting the proof of (4.2) in [H<sub>2</sub>]. With this, Lemma 5.2 can be restated thus.

**Lemma 5.3.** *There is a Lie group homomorphism  $\pi : \mathcal{H} \rightarrow \mathbb{R}$  and  $R > 0$  such that  $|\pi(p) - \pi(q)| < R$  for any two points  $p$  and  $q$  on the same leaf of  $\tilde{\mathcal{F}}_{bran}^{cu}$ .*

Almost all of the work in [H<sub>2</sub>] carries over immediately to the pointwise partially hyperbolic case. In particular, the results in Section 4 of that paper up to and including (4.8) also hold in our current setting. That section defines two projections  $\pi^s, \pi^u : \mathcal{H} \rightarrow \mathbb{R}$  which are Lie group homomorphisms, and from their special properties, we can show the following.

**Lemma 5.4.** *There is  $R > 0$  such that  $|\pi^s(p) - \pi^s(q)| < R$  for all  $p$  and  $q$  on the same leaf of  $\tilde{\mathcal{F}}_{bran}^{cu}$ .*

*Proof.* By (4.3) of [H<sub>2</sub>], there is a constant  $x_0 > 0$  such that  $|\pi^s(p)| < x_0$  implies  $|\pi^s \tilde{f}(p)| < x_0$  for all  $p \in \mathcal{H}$ . As a consequence, we can find a small unstable segment  $J$  such that  $|\pi^s(p)| < x_0$  for all  $n > 0$  and  $p \in \tilde{f}^n(J)$ . By (4.7), the diameter of  $\pi^u \tilde{f}^n(J)$  as a subset of  $\mathbb{R}$  tends to infinity as  $n \rightarrow \infty$ . Each of the curves  $\tilde{f}^n(J)$  is contained in a leaf of the lifted foliation  $\tilde{\mathcal{F}}_{bran}^{cu}$ . Further, it is not hard to show that the homomorphism  $\pi : \mathcal{H} \rightarrow \mathbb{R}$  given by Lemma 5.3 above can be written as a linear combination,  $\pi = a\pi^s + b\pi^u$ . The only way all of these estimates can hold is if  $b = 0$ . The result then follows.  $\square$

By repeating the arguments of Lemma 5.4 for  $f^{-1}$  in place of  $f$ , the following also holds.

**Lemma 5.5.** *There is  $R > 0$  such that  $|\pi^u(p) - \pi^u(q)| < R$  for all  $p$  and  $q$  on the same leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$ .*

This is a non-dynamically coherent analogue of (4.10) of [H<sub>2</sub>]. Using this version, we may repeat the proofs of (4.11), (4.12), and (4.13) of [H<sub>2</sub>], taking  $\mathcal{W}^{cs}(p)$  to mean *any* leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  which passes through the point  $p$ . We state the reformulations of (4.12) and (4.13) explicitly.

**Lemma 5.6.** *Every leaf of  $\tilde{\mathcal{F}}_{bran}^{cs}$  intersects every leaf of  $\tilde{\mathcal{W}}^u$  exactly once.*

**Lemma 5.7.** *For  $M > 0$  there is  $\ell > 0$  such that for any unstable curve  $J$  of length greater than  $\ell$ , the endpoints  $p$  and  $q$  satisfy  $|\pi^u(p) - \pi^u(q)| > M$ .*

*Proof of dynamical coherence.* We now prove dynamical coherence by showing that the “branching” foliation  $\tilde{\mathcal{F}}_{bran}^{cs}$  does not actually branch. Suppose instead that two distinct leaves  $L_1$  and  $L_2$  pass through a point  $p \in \mathcal{H}$  and take  $q_1 \in L_1 \setminus L_2$ . By Lemma 5.6, there is  $q_2 \in L_2 \cap \tilde{\mathcal{W}}^u(q_1)$ . By Lemma 5.7,  $|\pi^u \tilde{f}^n(q_1) - \pi^u \tilde{f}^n(q_2)| \rightarrow \infty$  as  $n \rightarrow \infty$ . By Lemma 5.5,  $|\pi^u \tilde{f}^n(q_1) - \pi^u \tilde{f}^n(q_2)| < 2R$  for all  $n$ , a contradiction.

This same argument allows us to show that there is a unique  $f$ -invariant foliation. If  $\mathcal{W}_1^{cs}$  and  $\mathcal{W}_2^{cs}$  are distinct  $f$ -invariant foliations, there are distinct leaves  $L_1$  and  $L_2$  from the lifted foliations which pass through a common point  $p \in \mathcal{H}$ . Applying Lemmas 5.7 and 5.5 as above again gives a contradiction.  $\square$

*Proof of leaf conjugacy.* As noted in [H<sub>2</sub>] (see the Remark after Lemma 4.10), only the proofs of (4.9) and (4.10) rely on absolute partial hyperbolicity. The first of these is dynamical coherence, proved above, and once dynamical coherence is established, (4.10) is equivalent to Lemma 5.5. As such, the entire proof of leaf conjugacy now holds in the pointwise case.  $\square$

**5.2. The general case.** The last subsection assumed the subbundles  $E^u$ ,  $E^c$  and  $E^s$  had orientations which were preserved by the derivative  $Df$  of  $f$ . Now consider the general case, where the assumption does not necessarily hold. For convenience, we simply say that a diffeomorphism preserves the orientation of a bundle if its derivative preserves the orientation.

**Proposition 5.8.** *If  $f : \mathcal{H}/\Gamma \rightarrow \mathcal{H}/\Gamma$  is partially hyperbolic, then  $E^u$ ,  $E^c$ , and  $E^s$  are orientable.*

*Proof.* Lift the subbundles to the universal cover  $\mathcal{H}$  and choose an orientation for each of them. Regarding  $\Gamma$  as the group of deck transformations, those  $\gamma \in \Gamma$  which preserve all three of these orientations form a normal, finite-index subgroup  $\Gamma_0$ . Lift  $f$  to  $\tilde{f} : \mathcal{H} \rightarrow \mathcal{H}$ . As  $\tilde{f}^2$  preserves the orientations of the subbundles, it descends to a diffeomorphism  $g$  on the nilmanifold  $\mathcal{H}/\Gamma_0$ . On this manifold, the subbundles are oriented and  $g$  preserves these orientations. Therefore, all of the previous analysis now applies. In particular, Lemma 5.7 holds for the unstable foliation on  $\mathcal{H}$ .

Choose an orientation for  $\mathbb{R}$ . Suppose  $x \in \mathcal{H}$  and that  $\alpha : \mathbb{R} \rightarrow \tilde{\mathcal{W}}^u(x)$  is an orientation-preserving parameterization of the unstable curve through  $x$ . Lemma 5.7 implies that  $\lim_{t \rightarrow +\infty} \pi^u \alpha(t)$  is either  $+\infty$  or  $-\infty$ . Basic continuity arguments show that this limit

does not depend on the choice of  $\alpha$  or  $x$ . It is straightforward to show that

$$\lim_{t \rightarrow +\infty} \pi^u \gamma \alpha(t) = \lim_{t \rightarrow +\infty} \pi^u \alpha(t)$$

for any deck transformation  $\gamma \in \Gamma$ . From this, it follows that the orientation of the lifted unstable bundle is  $\Gamma$ -invariant and descends to an orientation of  $E^u$  on  $\mathcal{H}/\Gamma$ . A similar argument shows that  $E^s$  is orientable. As the full three-dimensional tangent bundle  $E^u \oplus E^c \oplus E^s$  is orientable, the center bundle  $E^c$  is orientable as well.  $\square$

Though the subbundles are always orientable, there are examples of partially hyperbolic systems on  $\mathcal{H}/\Gamma$  which do not preserve these orientations. (To construct such an example, take a hyperbolic toral automorphism on  $\mathbb{T}^2$  which does not preserve the orientations of its stable and unstable subbundles. Then, build a bundle map over this.) Fortunately, this obstruction does not pose a serious problem.

Suppose  $f : \mathcal{H}/\Gamma \rightarrow \mathcal{H}/\Gamma$  is partially hyperbolic. Then,  $f^2$  preserves the orientations of the bundles  $E^u$ ,  $E^c$ , and  $E^s$ , and from the above proof of dynamical coherence, there is a unique  $f^2$ -invariant foliation  $\mathcal{W}$  tangent to  $E^{cs}$ . Note that  $f(\mathcal{W})$  is also an  $f^2$ -invariant foliation tangent to  $E^{cs}$  and so  $f(\mathcal{W}) = \mathcal{W}$ . Finding such an  $f$ -invariant foliation via Theorem 2.1 was the only reason for the added assumption on orientations, and therefore the results of the last subsection now follow in the general (non-orientation-preserving) case.

**5.3. Finite quotients.** To prove Theorem 1.1 from Theorem 1.2 we must consider finite quotients of nilmanifolds.

**Proposition 5.9.** *If a compact manifold  $M$  has a universal cover homeomorphic to  $\mathbb{R}^3$  and  $\pi_1(M)$  is virtually nilpotent, then there is a regular finite cover  $\hat{M}$  over  $M$  such that  $\hat{M}$  is diffeomorphic to a nilmanifold and every diffeomorphism  $f : M \rightarrow M$  can be lifted to a diffeomorphism  $\hat{f} : \hat{M} \rightarrow \hat{M}$ .*

*Proof.* As  $\pi_1(M)$  is virtually nilpotent, the Hirsch-Plotkin radical  $N$  is the maximal normal nilpotent subgroup and is of finite index. It defines a regular finite covering  $\hat{M}$ . As any automorphism of  $\pi_1(M)$  leaves  $N$  invariant, any diffeomorphism of  $M$  lifts to  $\hat{M}$ . By a classical result of P. A. Smith, any free action on  $\mathbb{R}^n$  is torsion free (see [Bor, page 43]), and then by a result of Malcev [Mal],  $N$  can be identified with the fundamental group of a compact nilmanifold  $\hat{K}$ . As  $\hat{M}$  and  $\hat{K}$  are aspherical and have isomorphic fundamental groups, they are homotopy equivalent and  $\hat{K}$  must be three-dimensional. One can verify directly that three-dimensional nilmanifolds are “sufficiently large” in the sense of Waldhausen, and therefore,  $\hat{M}$  and  $\hat{K}$  are in fact homeomorphic [Wal]. By a result of Moise, they are diffeomorphic [Moi].  $\square$

*Proof of Theorem 1.1.* If  $f : M \rightarrow M$  is as in Theorem 1.1, it lifts to  $\hat{f} : \hat{M} \rightarrow \hat{M}$  by the above proposition, and by Theorem 1.2 there is a foliation  $\mathcal{W}$  which for each  $k \geq 1$  is the unique  $\hat{f}^k$ -invariant foliation tangent to  $E^{cs}$ . For each deck transformation  $\alpha$  of the finite covering of  $\hat{M}$  over  $M$ , there is  $k$  such that  $\hat{f}^k \alpha = \alpha \hat{f}^k$ , so that  $\alpha(\mathcal{W})$  is also  $\hat{f}^k$ -invariant, and by uniqueness  $\alpha(\mathcal{W}) = \mathcal{W}$ . As such,  $\mathcal{W}$  quotients to a foliation on  $M$  and is the unique  $f$ -invariant foliation tangent to  $E^{cs}$ .  $\square$

## 6. DYNAMICAL CONSEQUENCES

**6.1. Entropy maximizing measures.** In this subsection, we observe how our results allow one to re-obtain some results about entropy maximizing measures previously proven under less generality. First, the main result of [U] also holds in the pointwise case.

**Proposition 6.1.** *Let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a partially hyperbolic diffeomorphism such that its linear part  $A_f$  is hyperbolic. Then,  $f$  has a unique maximal entropy measure which is measurably equivalent to the volume measure for  $A_f$  (i.e. it is intrinsically ergodic).*

*Proof.* The only dependence on absolute partial hyperbolicity in [U] is in establishing Lemmas 3.2 and 3.3. Those lemmas can be proven directly from Corollary 1.5 of this paper.  $\square$

In higher dimensions, under the assumption of being isotopic to Anosov along a path of partially hyperbolic systems it is possible to recover also the same result in the pointwise case ([FPS]).

**Proposition 6.2.** *Let  $f$  be a  $C^{1+\alpha}$  partially hyperbolic diffeomorphism of a non-toral three-dimensional nilmanifold. Then,  $f$  has at least one and at most a finite number of measures of maximal entropy.*

*Proof.* By Theorem 1.2,  $f$  has a center foliation of compact leaves, and is also accessible (see Proposition 6.4 below). The proposition then follows immediately from the main result of [RHRHTU]. See that paper for more details.  $\square$

Under additional assumptions, one can prove similar results for diffeomorphisms on  $\mathbb{T}^3$  with non-hyperbolic linear part. We again refer the interested reader to [RHRHTU].

**6.2. Uniqueness of attractors on nilmanifolds.** In this section we deduce some dynamical consequences from the main results to the study of dynamics of partially hyperbolic diffeomorphisms in 3 dimensional non-toral nilmanifolds.

We recall that a *quasi-attractor* is a chain-recurrence class<sup>2</sup> such that it admits a basis of neighborhoods  $\{U_n\}$  such that  $f(\overline{U_n}) \subset U_n$ . It is well known (and simple to show) that

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<sup>2</sup>See [BDV] Chapter 10.

a quasi-attractor must be saturated by unstable sets. Thus, in the partially hyperbolic setting saturated by the unstable foliation.

As a consequence of an argument very similar to the one in [BCLJ] (Proposition 4.2) we can deduce that for a partially hyperbolic diffeomorphism of  $\mathcal{H}/\Gamma$  these foliation have a unique minimal set, thus concluding:

**Proposition 6.3.** *Let  $f : \mathcal{H}/\Gamma \rightarrow \mathcal{H}/\Gamma$  be a strong partially hyperbolic diffeomorphism, then,  $f$  has a unique quasi-attractor. Moreover, there exists a leaf of  $\mathcal{W}^u$  which is dense in  $\mathcal{H}/\Gamma$  and a unique minimal set of the foliation.*

One could wonder if such a result does not imply that every strong partially hyperbolic diffeomorphism on  $\mathcal{H}/\Gamma$  is transitive, however, this result is sharp: Y. Shi has constructed examples of partially hyperbolic diffeomorphisms in nilmanifold which are Axiom A (in particular they have an attractor and a repeller). The construction is in the spirit of [BG].

*Proof.* The key point is that this problem can be reduced to a problem about dynamics of homeomorphisms of  $\mathbb{T}^2$  which are skew-products over the irrational rotation. In fact, we can consider a two-dimensional torus  $T$  consisting of center leaves and transverse to both the strong stable and strong unstable foliations of  $\Phi_f$ , the algebraic part of  $f$ .

We consider the leaf conjugacy  $h : \mathcal{H}/\Gamma \rightarrow \mathcal{H}/\Gamma$  from  $f$  to  $\Phi_f$ .

We obtain that  $h^{-1}(T)$  is a topological torus, foliated by center leaves and such that it is transverse to both strong foliations for  $f$ . Now, the return map of the unstable holonomy defines a homeomorphism of  $T$  which after conjugacy can be written in the following form:

$$F(x, y) = (x + \alpha, \varphi_x(y)) \mod \mathbb{Z}^2$$

and satisfying that  $\varphi_x(y + 1) = \varphi_x(y) + 1$  and that  $\varphi_{x+1}(y) = \varphi_x(y) + k$  (this is to say that  $F$  is homotopic to a Dehn twist).

We will divide the proof in two claims about such homeomorphisms. Notice that the existence of a unique minimal set for the unstable foliation implies immediately that  $f$  has a unique quasi-attractor since quasi-attractors are compact disjoint and saturated by the unstable foliation.

The following claim<sup>3</sup> implies that  $f$  has a dense unstable leaf:

**Claim.**  *$F$  is transitive.*

*Proof.* Consider two open sets  $U_1, U_2$  in  $\mathbb{T}^2$ . We will show that there is an iterate of  $U$  that intersects  $V$ . To do this, we consider arcs  $J_i \subset U_i$  of the form  $J_i = (x_i^-, x_i^+) \times \{y_i\}$ .

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<sup>3</sup>We thank Tobias Jäger who communicated to us this argument.



We will work in the universal cover  $\mathbb{R}^2$  of  $\mathbb{T}^2$ . We claim that if we denote as  $\tilde{F}$  to the lift of  $F$  and we consider the iterates  $\tilde{J}_1^k = \tilde{F}^k(\tilde{J}_1)$  and we define  $\ell_k$  to be the length of  $\tilde{J}_1^k$  when projected in the second coordinate, then  $\ell_k$  goes to infinity with  $k$ . Now, since the dynamics on the base is minimal this implies that there is an iterate of  $F$  such that  $F(J_1) \cap J_2 \neq \emptyset$  which will conclude the proof.

To prove the claim, consider a sequence of iterates  $J_1, f^{i_1}(J_1), \dots, f^{i_m}(J_1)$  such that the projection of their union in the first coordinate is surjective. It is clear that this can be done since the dynamics in the first coordinate is minimal. Now, join these arcs by arcs contained in the second coordinate in order to make a non-trivial loop  $\gamma$  homotopic to the first coordinate circle. The iterates of  $\gamma$  by  $F$  start to turn around the second coordinate infinitely many times and thus the length of  $F^n(\gamma)$  goes to infinity. since the vertical arcs remain of bounded length, we conclude the claim.  $\diamond$

The following claim concludes the proof of the Proposition.

**Claim.**  *$F$  has a unique minimal set.*

*Proof.* We follow the argument in Proposition 4.2 of [BCLJ] which with the use of the previous proposition allows to conclude.

Assume there are two disjoint minimal sets  $K, K'$  in  $\mathbb{T}^2$ . We know that  $K \cap K' = \emptyset$  and since they are compact, they are at bounded distance from below. Consider the set  $U_1$  as the set of (ordered) arcs  $\{x\} \times (y_1, y_2)$  with  $y_1 \in K$  and  $y_2 \in K'$ . Similarly one can consider the set of arcs  $U_2$  of the form  $\{x\} \times (y_1, y_2)$  with  $y_1 \in K'$  and  $y_2 \in K$ . Clearly,  $U_1 \cap U_2 = \emptyset$ , and since the maps  $\varphi_x$  are order preserving they are invariant. Both sets intersect every fiber of the type  $\{x\} \times S^1$  since the base dynamics is minimal.

We will prove that both have non-empty interior which will contradict transitivity. To do this, notice that compactness of  $K$  and  $K'$  implies that if we consider the mappings  $x \mapsto (K \cap \{x\} \times S^1)$  and  $x \mapsto (K' \cap \{x\} \times S^1)$  which are both semi-continuous and thus share a residual set of continuity points. Take  $x$  a common continuity point, then any point of the form  $(x, y)$  in  $U_i$  is an interior point of  $U_i$  concluding the proof.  $\diamond \square$

**6.3. Accessibility.** A partially hyperbolic diffeomorphism  $f : M \rightarrow M$  is *accessible* if any two points  $x, y \in M$  can be connected by a concatenation of paths, each path tangent either to  $E^u$  or  $E^s$ . As with dynamical coherence, we can now show there are manifolds where every pointwise partially hyperbolic system is accessible.

**Proposition 6.4.** *Suppose  $M$  is a non-toral 3-dimensional nilmanifold. Every  $C^1$  partially hyperbolic diffeomorphism on  $M$  (measure-preserving or not) is accessible.*

This was previously proven in the measure-preserving case in [RHRHU<sub>2</sub>].

*Proof.* If  $f$  is not accessible, there is a non-empty lamination  $\Lambda \subset M$  whose leaves are complete and tangent to  $E^u \oplus E^s$  [RHRHU<sub>1</sub>, §3]. Lift  $\Lambda$  to the universal cover  $\mathcal{H}$  and choose a leaf  $S$  of the lifted lamination. By Global Product Structure, as proved in [H<sub>2</sub>] and now extended to the pointwise case, such a surface  $S$  intersects each center leaf exactly once. For any deck transformation  $\gamma \in \Gamma$ , the surface  $\gamma(S)$  also intersects each center leaf exactly once. Further  $\gamma(S)$  and  $S$  are either disjoint or coincide. By Proposition 5.8,  $E^c$  is orientable on  $\mathcal{H}/\Gamma$ , and so there is a  $\Gamma$ -invariant orientation of  $E^c$  on  $\mathcal{H}$ . This orientation allows us to define an ordering on the surfaces  $\gamma(S)$  for  $\gamma \in \Gamma$ , saying whether one such surface is “above” or “below” another. From Proposition A.7 in Appendix A, this ordering must be trivial. However, as shown in [H<sub>2</sub>] there are elements  $\gamma \in \Gamma$  which fix each center leaf, but do not fix any point of  $\mathcal{H}$ . This gives a contradiction.  $\square$

## APPENDIX A. CLASSIFICATION OF REEBLESS FOLIATIONS IN NILMANIFOLDS

In this section, a *leaf system* is a tuple  $(M, \hat{M}, \mathcal{F}, S)$  where

- $M$  is a compact manifold without boundary,
- $\hat{M}$  is a normal (or regular) covering space of  $M$ ,
- $\mathcal{F}$  is a transversely oriented foliation on  $M$ ,
- $S$  is a leaf of  $\hat{\mathcal{F}}$ , the foliation obtained by lifting  $\mathcal{F}$  to  $\hat{M}$ , and
- there is no closed loop topologically transverse to  $\hat{\mathcal{F}}$  passing through  $S$ .

While the last condition implies that  $S$  is properly embedded in  $\hat{M}$ , other leaves of  $\hat{\mathcal{F}}$  may not be. Also note that  $\hat{M}$  is not necessarily the universal cover of  $M$ . Assume such a leaf system is fixed, and let  $L$  denote the group of deck transformations on  $\hat{M}$ . If  $\gamma \in L$ , let  $x\gamma$  denote the action of  $\gamma$  on  $x \in \hat{M}$ . A right action was chosen to keep the notation as close as possible to that used in [BBI<sub>2</sub>].  $S$  splits  $\hat{M}$  into two open subspaces,  $S_+$  and  $S_-$  where the sign is given by the transverse orientation on  $\mathcal{F}$ . Define

$$\begin{aligned}\Gamma_+ &= \{\gamma \in L : S_+\gamma \subset S_+\}, \\ \Gamma_- &= \{\gamma \in L : S_+\gamma \supset S_+\}, \\ \Gamma &= \Gamma_+ \cup \Gamma_-.\end{aligned}$$

**Lemma A.1** (confer Lemma 3.9 of [BBI<sub>2</sub>]). *If  $A$  is an abelian subgroup of  $L$ , then  $A \cap \Gamma$  is a subgroup of  $A$ .*

For  $X \subset \hat{M}$  and  $H \subset L$ , define the notation  $XH := \{x\gamma : x \in \hat{M}, \gamma \in H\}$ .

**Lemma A.2** (confer Lemma 3.11 of [BBI<sub>2</sub>]). *If  $A$  is an abelian subgroup of  $L$  and  $S_+A = S_-A = \hat{M}$  then  $A \subset \Gamma$ .*

**Lemma A.3.** *Suppose  $(M, \tilde{M}, \mathcal{F}, S)$  is a leaf system, where  $\tilde{M}$  is the universal cover of  $M$ ,  $N$  is a normal subgroup of the group of deck transformations and  $\tilde{U} := S_+N \neq \tilde{M}$ . Then, for  $\hat{M} = \tilde{M}/N$  and  $\hat{U} = \tilde{U}/N$ , and any connected component  $C$  of the (non-empty) boundary of  $\hat{U}$ , the tuple  $(M, \hat{M}, \mathcal{F}, C)$  is a leaf system.*

*Proof.* As  $\tilde{U}$  is  $N$ -invariant, it is easy to see that  $\emptyset \neq \hat{U} \neq \hat{M}$  and that  $\hat{U}$  is open and is a union of leaves of the lift  $\hat{\mathcal{F}}$  of the foliation  $\mathcal{F}$ . Further, while the boundary of  $\hat{U}$  may have several components, each has the same orientation, and so any curve transverse to  $\hat{\mathcal{F}}$  which exits  $\hat{U}$ , cannot later re-enter  $\hat{U}$ . This shows the last item in the definition of “leaf system” and the others are immediate.  $\square$

**Lemma A.4** (confer Lemma 3.12 of [BBI<sub>2</sub>]). *Suppose the leaf system is such that there is a group isomorphism  $h : L \rightarrow \mathbb{Z}^d$  and  $S_+L = S_-L = \hat{M}$ . Then,  $\Gamma = L$  and there is a hyperplane  $P$  separating  $\mathbb{R}^d$  into closed half-spaces  $H_+$  and  $H_-$  such that  $h(\Gamma_+) \subset H_+$  and  $h(\Gamma_-) \subset H_-$ .*

All of the results listed so far hold for any  $C^0$  foliation  $\mathcal{F}$ . To proceed further, we need to add an assumption of “uniform structure.”

**Assumption.** There are constants  $r, R > 0$  such that if  $x \in S_+$  then there is  $y \in \hat{M}$  such that  $B_r(y) \subset B_R(x) \cap S_+$ . Similarly, if  $x \in S_-$ , there is  $y \in \hat{M}$  such that  $B_r(y) \subset B_R(x) \cap S_-$ .

This property holds for any foliation tangent to a  $C^0$  subbundle of the tangent bundle, so long as the base space  $M$  is compact.

**Lemma A.5.** *If  $X = \{x_1, \dots, x_n\}$  is an  $r$ -net of a fundamental domain of  $\hat{M}$ , then  $XL \cap S_+$  is an  $R$ -net for  $S_+$ . Further, if  $\Gamma = L$  and  $X \cap S_+ = \emptyset$ , then  $X\Gamma_+$  is an  $R$ -net for  $S_+$ .*

*Proof.* The first part follows immediately from the above assumption and the fact that  $XL$  is an  $r$ -net for all of  $\hat{M}$ . To prove the second part, suppose  $y \in S_+$ . By the first part, there is  $x_i \in X$  and  $\gamma \in L$  such that  $y \in B_R(x_i\gamma)$  and  $x_i\gamma \in S_+$ . Then,  $x_i$  demonstrates that  $S_+\gamma^{-1}$  is not a subset of  $S_+$ , which, in the special case of  $L = \Gamma = \Gamma_+ \cup \Gamma_-$  implies that  $\gamma \in \Gamma_+$ .  $\square$

We say that a codimension one manifold  $P$  has the same “ordering” as  $S$  if  $P$  splits  $\hat{M}$  into two subspaces  $P_+$  and  $P_-$  where  $P_+\gamma \subset P_+$  for all  $\gamma \in \Gamma_+$  and  $P_+\gamma \supset P_+$  for all  $\gamma \in \Gamma_-$ .

**Proposition A.6.** *Suppose  $\Gamma = L$  and  $S_+\Gamma = S_-\Gamma = \hat{M}$  and that there is a codimension one manifold  $P$  with the same “ordering” as  $S$ . Then  $S$  lies a finite distance from  $P$ .*

*Proof.* Since  $S_- \Gamma = \hat{M}$ , we can find a fundamental domain for  $\hat{M}$  inside of  $S_-$ . Let  $X = \{x_1, \dots, x_n\}$  be an  $r$ -net for this fundamental domain, where  $r$  is as in Lemma A.5. Fix a point  $y \in P_+$  and set  $D = \sup_i d(x_i, y)$  where  $d$  is distance measured on  $\hat{M}$ . Then, for  $\gamma \in \Gamma_+$ ,  $d(x_i \gamma, y \gamma) \leq D$ , and  $x_i \gamma \in P_+ \gamma \subset P_+$ . This means that every point of  $X \Gamma_+$ , is at distance at most  $D$  from  $P_+$ . By Lemma A.5,  $X \Gamma_+$  is an  $R$ -net for  $S_+$  and so every point in  $S_+$  is at distance at most  $R + D$  from  $P_+$ . A similar argument shows that every point in  $S_-$  is a bounded distance from  $P_-$  and completes the proof.  $\square$

Now consider the specific case of the Heisenberg group.

**Proposition A.7.** *If a relation “ $\leq$ ” on the Heisenberg lattice*

$$\Gamma = \langle x, y, z : xy = yxz, xz = zx, yz = zy \rangle$$

*satisfies all of the following properties*

- *reflexivity:*  $u \leq u$ ,
- *totality:* either  $u \leq v$  or  $v \leq u$  or both,
- *transitivity:*  $u \leq v$  and  $v \leq w$  implies  $u \leq w$ ,
- *right-invariance:*  $u \leq v$  implies  $uw \leq vw$ , and
- *an “Archimedean” property for  $z$ :* for all  $u \in \Gamma$ , there is  $k(u) \in \mathbb{Z}$  such that

$$z^{k(u)} \leq u \leq z^{k(u)+1},$$

*then  $\leq$  is trivial:*  $u \leq v \leq u$  for all  $u$  and  $v$ .

*Proof.* Using the Archimedean property, let  $i, j \in \mathbb{Z}$  be such that  $z^i \leq x \leq z^{i+1}$  and  $z^j \leq y \leq z^{j+1}$ . Since  $z$  commutes with both  $x$  and  $y$ , for any  $n \in \mathbb{Z}$

$$z^{(i+j)n} \leq x^n y^n \leq z^{(i+j+2)n}$$

and

$$\begin{aligned} z^{(i+j)n} &\leq y^n x^n \leq z^{(i+j+2)n} \Rightarrow \\ z^{(i+j)n+n^2} &\leq y^n x^n z^{n^2} \leq z^{(i+j+2)n+n^2}. \end{aligned}$$

Notice that  $x^n y^n = y^n x^n z^{n^2}$  and therefore

$$z^{(i+j)n+n^2} \leq z^{(i+j+2)n} \Rightarrow z^{n^2} \leq z^{2n}$$

for all  $n \in \mathbb{Z}$ . This is enough to deduce  $z^{k_1} \leq z^{k_2}$  for any two integers  $k_1, k_2$ , and then by the Archimedean property

$$u \leq z^{k(u)+1} \leq z^{k(v)} \leq v$$

for any  $u$  and  $v$ , showing that the relation is trivial.  $\square$

Any three-dimensional nilmanifold can be thought of as a circle bundle over a 2-torus. If the bundle is trivial, the manifold is the 3-torus. If it is non-trivial, then it can be finitely covered by a manifold  $M$  whose fundamental group is exactly as in Property A.7. Therefore, we will only consider this specific case  $M$ . The base space  $\mathbb{T}^2$  has universal cover  $\mathbb{R}^2$ . Therefore, the bundle  $M$  is covered by a circle bundle  $\hat{M}$  whose base space is  $\mathbb{R}^2$ . Both  $M$  and  $\hat{M}$  have as universal cover the Heisenberg group, here denoted by  $\tilde{M}$ . For a Lie group homomorphism  $\pi : \tilde{M} \rightarrow \mathbb{R}$ , recall the definition of the foliation  $\mathcal{F}_\pi$  where each leaf of  $\tilde{\mathcal{F}}_\pi$  is a level-set of  $\pi$ .

**Theorem A.8.** *If  $M$  is a non-toral three-dimensional nilmanifold, and  $\mathcal{F}$  is a  $C^0$  Reebless foliation on  $M$  with uniform structure (as explained above), then  $\mathcal{F}$  is almost aligned with  $\mathcal{F}_\pi$  for some  $\pi$ .*

*Proof.* Without loss of generality, we may replace  $M$  by a finite cover, such that the fundamental group of  $M$  is exactly the group  $\Gamma$  given in Proposition A.7. As  $\mathcal{F}$  is Reebless, there are no closed cycles transverse to the lifted foliation  $\tilde{\mathcal{F}}$ . Fix a lifted leaf  $S$ . This defines a leaf system  $(M, \tilde{M}, \mathcal{F}, S)$ . As before, let  $L$  denote the set of deck transformations on  $\tilde{M}$ . Let  $Z \subset L$  be the center of this group.

We first assume that  $S_+Z = S_-Z = \tilde{M}$  and show this leads to a contradiction. Under this assumption, pick any  $\gamma \in L$  and define  $A_\gamma$  as the smallest subgroup of  $L$  containing  $Z$  and  $\gamma$ . As this group is abelian and  $S_+A_\gamma = S_-A_\gamma = \tilde{M}$ , Lemma A.2 implies that  $A_\gamma \subset \Gamma$ . As  $\gamma$  was chosen arbitrarily, this shows that  $\Gamma = L$ .

The center  $Z$  is cyclic, and we may take a generator  $z$  such that  $S_+ \subset S_+z$ . Consider  $\gamma \in L$  and take points  $x \in S_+\gamma$  and  $y \in \tilde{M} \setminus S_+\gamma$ . By the assumption  $S_+Z = S_-Z = \tilde{M}$ , there are integers  $i < j$  such that  $x \in \tilde{M} \setminus S_+z^i$  and  $y \in S_+z^j$ . Using that  $L = \Gamma = \Gamma_+ \cup \Gamma_-$ ,

$$S_+z^i \subset S_+\gamma \subset S_+z^j.$$

Hence, there is  $k \in \mathbb{Z}$ ,  $i \leq k < j$  such that

$$S_+z^k \subset S_+\gamma \subset S_+z^{k+1}.$$

Define a relation “ $\leq$ ” on  $\Gamma = L$  by  $\alpha \leq \beta$  if  $S_+\alpha \subset S_+\beta$ . This relation satisfies the hypotheses of Proposition A.7 and is therefore trivial:  $S_+\gamma = S_+$  for all  $\gamma \in \Gamma$ . This contradicts the assumption  $S_+Z = \tilde{M}$ .

We have reduced to the case where either  $S_+Z \neq \tilde{M}$  or  $S_-Z \neq \tilde{M}$ . Without loss of generality, assume the first. Consider now a new leaf system  $(M, \hat{M}, \mathcal{F}, C)$  given by Lemma A.3. The group of deck transformations  $\hat{L} = L/Z$  is isomorphic to  $\mathbb{Z}^2$ .

First consider the case  $C_+\hat{L} \neq \hat{M}$ . The boundary of  $C_+\hat{L}$  quotients to a union of compact leaves of  $M$ . Let  $T$  be such one such leaf. As  $\mathcal{F}$  is Reebless, the embedding  $T \hookrightarrow M$  is  $\pi_1$ -injective, and this implies that  $T$  is either a 2-sphere, or a 2-torus. The

Reeb stability theorem rules out the possibility of a sphere. Then, as a  $\pi_1$ -injective torus,  $T$  lifts to a cylinder  $\hat{T} \subset \hat{L}$  and one can verify  $\hat{T}_+ \hat{L} = \hat{T}_- \hat{L} = \hat{M}$ . Therefore, up to replacing the leaf  $C$ , we may freely assume that  $C_+ \hat{L} = C_- \hat{L} = \hat{M}$ .

Applying Lemma A.4 and Proposition A.6, the leaf  $C$  is a finite distance away from a surface  $\hat{P}$  which is the pre-image by the projection  $\hat{M} \rightarrow \mathbb{R}^2$  of a geometric line on  $\mathbb{R}^2$ .

Fix a fundamental domain  $K$  of the covering  $\hat{M} \rightarrow M$ . Since  $C_+ \hat{L} = C_- \hat{L} = \hat{M}$ ,  $K$  must lie inside a set  $U := C_+ \alpha \cap C_- \beta$  for some  $\alpha, \beta \in \hat{L}$ . Further, there is  $D > 0$  such that every point of  $U$  is at distance at most  $D$  from  $\hat{P}$ . Since there are no topological crossings, any leaf  $C'$  of  $\hat{\mathcal{F}}$  passing through  $K$  must lie in the closure of  $U$  and so  $C'$  lies at most a distance  $D$  away from  $\hat{P}$ . By considering translates  $C'\gamma$  for  $\gamma \in \hat{L}$ , we can show that every leaf of  $\hat{\mathcal{F}}$  lies at most a distance  $D$  away from some translate  $\hat{P}\gamma$  of  $\hat{P}$ . There is a Lie group homomorphism  $\pi : \tilde{M} \rightarrow \mathbb{R}$  such that each translated surface  $\hat{P}\gamma$  lifts to a leaf of  $\tilde{\mathcal{F}}_\pi$ . Then, lifting leaves of  $\hat{\mathcal{F}}$  to leaves of  $\tilde{\mathcal{F}}$ , the result is proved.  $\square$

## APPENDIX B. CLASSIFICATION OF FOLIATIONS IN TORUS BUNDLES OVER THE CIRCLE

In this section we give a classification result of foliations of 3-dimensional manifolds which are torus bundles over the circle  $S^1$ . This result applied to mapping torus of Anosov automorphisms will be also used in [HP].

We will classify Reebless foliations of such manifolds under the relation of being almost aligned with some model foliation. In certain cases, namely, when there are no torus leaves, we will be able to obtain a stronger relation.

Two branching foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are *almost parallel* if there exists  $R > 0$  such that:

- For every leaf  $L_1 \in \tilde{\mathcal{F}}_1$  there exists a leaf  $L_2 \in \tilde{\mathcal{F}}_2$  such that  $L_1 \subset B_R(L_2)$  and  $L_2 \subset B_R(L_1)$  (i.e. the Hausdorff distance between  $L_1$  and  $L_2$  is smaller than  $R$ ).
- For every leaf  $L_2 \in \tilde{\mathcal{F}}_2$  there exists a leaf  $L_1 \in \tilde{\mathcal{F}}_1$  such that  $L_1 \subset B_R(L_2)$  and  $L_2 \subset B_R(L_1)$ .

Foliations  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_4$  in Figure 1 are almost parallel to each other, however,  $\mathcal{F}_3$  is not almost parallel to them.

We state the definition for branching foliations since in fact, the results of [BI] give that the branching foliations they construct are almost parallel to some Reebless foliation. Moreover, we have the following:

**Proposition B.1.** *The following properties are verified:*

- (i) *Being almost parallel is an equivalence relation among branching foliations.*

- (ii) *If  $\mathcal{W}$  is almost aligned with  $\mathcal{W}'$  a foliation in  $M$  and  $\varphi$  a diffeomorphism of  $M$  isotopic to the identity, then  $\varphi(\mathcal{W})$  is almost parallel to  $\mathcal{W}$  and almost aligned to  $\mathcal{W}'$ .*

PROOF. Property (i) follows from the triangle inequality. Properties (ii) follows from the fact that the map in the universal cover is at bounded distance from the identity.

□

For  $C^2$ -foliations, Plante (see [Pl]) gave a classification of foliations without torus leaves in 3-dimensional manifolds with almost solvable fundamental group. His proof relies on the application of a result from [Rou] which uses the  $C^2$ -hypothesis in an important way (other results which used the  $C^2$ -hypothesis such as Novikov's Theorem are now well known to work for  $C^0$ -foliations thanks to [So]). We shall use a recent result of Gabai [Ga] which plays the role of Roussarie's result and allows the argument of Plante to be recovered.

We state now a consequence of Theorem 2.7 of [Ga] which will serve our purposes<sup>4</sup>:

**Theorem B.2.** *Let  $\mathcal{F}$  be a foliation of a 3-dimensional manifold  $M$  without closed leaves and let  $T$  be an embedded two-dimensional torus whose fundamental group injects in the one of  $M$ , then,  $T$  is isotopic to a torus which is transverse to  $\mathcal{F}$ .*

On the one hand, Gabai proves that a closed incompressible surface is homotopic to a surface which is either a leaf of  $\mathcal{F}$  or intersects  $\mathcal{F}$  only in isolated saddle tangencies. Since the torus has zero Euler characteristic, this implies that it must be transverse to  $\mathcal{F}$ . We remark that Gabai's result is stated by the existence of a homotopy, and this must be so since Gabai starts with an *immersed* surface, however, it can be seen that Lemma 2.6 of [Ga] can be done by isotopies if the initial surface is embedded. The rest of the proof uses only isotopies. See also [Cal] Lemma 5.11 and the Remark after Corollary 5.13.

In view of this result and in order to classify foliations in torus bundles over the circle it is natural to look at foliations of  $\mathbb{T}^2 \times [0, 1]$ . By considering a gluing of the boundaries by the identity map, we get a foliation of  $\mathbb{T}^3$ . These foliations (in the  $C^0$ -case) were classified in [Pot<sub>1</sub>], and the result can be restated in the terms used in this paper as follows:

**Theorem B.3** (Theorem 5.4 and Proposition 5.7 of [Pot<sub>1</sub>]). *Let  $\mathcal{W}$  be a Reebless foliation of  $\mathbb{T}^3$ , then,  $\mathcal{W}$  is almost aligned with a linear foliation of  $\mathbb{T}^3$ . Moreover, if the linear foliation is not a foliation by tori, then  $\mathcal{W}$  is almost parallel to the linear foliation and if it is a foliation by tori then there is at least one torus leaf.*

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<sup>4</sup>Notice that a foliation of a 3-dimensional manifold without closed leaves is *taut*, see [Cal] Chapter 4 for definitions and these results.

A *linear foliation* of  $\mathbb{T}^3$  is the projection by the natural projection  $p : \mathbb{R}^3 \rightarrow \mathbb{R}^3/\mathbb{Z}^3 \cong \mathbb{T}^3$  of a linear foliation of  $\mathbb{R}^3$ . It is a foliation by tori if the linear foliation is given by a plane generated by two vectors in  $\mathbb{Z}^3$ . The same classification can be done for one-dimensional foliations of  $\mathbb{T}^2$  for which the proof is easier (see for example section 4.A of [Pot<sub>2</sub>]). This allows us to classify foliations of  $\mathbb{T}^2 \times [0, 1]$  transverse to the boundary:

**Proposition B.4.** *Let  $\mathcal{W}$  be a foliation of  $\mathbb{T}^2 \times [0, 1]$  which is transverse to the boundary and has no torus leaves. Then, the foliation  $\mathcal{W}$  is almost aligned with a foliation of the form  $\mathcal{L} \times [0, 1]$  where  $\mathcal{L}$  is a linear foliation of  $\mathbb{T}^2$ . If  $\mathcal{L}$  is not a foliation by circles, then  $\mathcal{W}$  is almost parallel to  $\mathcal{L} \times [0, 1]$ .*

*Proof.* The proof can be done directly (see [Rou, Pl] for the  $C^2$ -case). We will use Theorem B.3 instead. Consider the foliation  $\mathcal{W}'$  of  $\mathbb{T}^2 \times [0, 2]$  obtained by gluing  $\mathbb{T}^2 \times [0, 1]$  with the foliation  $\mathcal{W}$  with  $\mathbb{T}^2 \times [1, 2]$  with the foliation  $\varphi(\mathcal{W})$  where  $\varphi_1 : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2 \times [1, 2]$  is given by  $\varphi_1(x, t) = (x, 2 - t)$ . It is not hard to check that this gives rise to a well defined foliation  $\mathcal{W}'$  of  $\mathbb{T}^2 \times [0, 2]$  (is like putting a mirror in the torus  $\mathbb{T}^2 \times \{1\}$ ).

We can now construct a foliation of  $\mathbb{T}^3$  as follows: we glue  $\mathbb{T}^2 \times \{0\}$  with  $\mathbb{T}^2 \times \{2\}$  by the diffeomorphism  $\varphi_2 : \mathbb{T}^2 \times \{0\} \rightarrow \mathbb{T}^2 \times \{2\}$  given by  $\varphi_2(x, 0) = (x, 2)$ . Again, it is easy to show that the foliation can be defined in  $\mathbb{T}^3 = \mathbb{T}^2 \times [0, 2]/\varphi_2$ .

By the previous Theorem, we know that the resulting foliation is almost aligned with a linear foliation of  $\mathbb{T}^3$ . Since we have assumed that there is no torus leaves of  $\mathcal{W}$  we know that this linear foliation cannot be the one given by the planes  $\mathbb{R}^2 \times \{t\}$  so, it must be transverse to the boundaries of  $\mathbb{T}^2 \times [0, 1]$ . This concludes the proof.  $\square$

**Remark.** As a consequence of the previous result we get the following: The foliations  $\mathcal{W} \cap (\mathbb{T}^2 \times \{0\})$  and  $\mathcal{W} \cap (\mathbb{T}^2 \times \{1\})$  are almost aligned with each other. In particular, one can prove that if in one of the boundary components is almost parallel to a linear foliation, then the whole foliation  $\mathcal{W}$  is almost parallel to a linear foliation of  $\mathbb{T}^2$  times  $[0, 1]$ .

Consider the manifold  $M_\psi$  obtained by  $\mathbb{T}^2 \times [0, 1]$  by identifying  $\mathbb{T}^2 \times \{0\}$  with  $\mathbb{T}^2 \times \{1\}$  by a diffeomorphism  $\psi$ . Let  $p : M_\psi \rightarrow S^1 = [0, 1]/\sim$  given by the projection in the second coordinate. Any torus bundle over the circle can be constructed this way, naturally, if  $\psi$  and  $\psi'$  are isotopic then  $M_\psi$  and  $M_{\psi'}$  are diffeomorphic.

The construction of  $M_\psi$  determines a incompressible torus in  $M_\psi$  which we will assume remains fixed. Under this choice of incompressible torus we can consider a family of foliations of  $M_\psi$  transverse to such torus.

We are now able to classify foliations in torus bundles over  $S^1$  depending on the isotopy class of  $\psi$ .



Any manifold of the form  $\mathcal{H}/\Gamma$  can be constructed as a torus bundle over  $S^1$ : The monodromy being given by (something isotopic to) the diffeomorphism  $\psi_k : \mathbb{R}^2/\mathbb{Z}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$  given by:

$$\psi_k(x) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} x \quad \text{mod } \mathbb{Z}^2$$

In the case that  $\psi : \mathbb{T}^2 \cong S^1 \times S^1 \rightarrow \mathbb{T}^2$  is a Dehn-twist of the form:

$$\psi(t, s) = (t, s + kt) \pmod{\mathbb{Z}^2}$$

$M_\psi$  is homeomorphic to a nilmanifold of the form  $\mathcal{H}/\Gamma$ . We define the foliations  $\mathcal{F}_\theta$  on  $M_\psi$  given by starting with the linear foliation  $\mathcal{L}$  of  $\mathbb{T}^2$  by circles of the form  $\{t\} \times S^1$  and we consider the foliation  $\mathcal{L} \times [0, 1]$  of  $\mathbb{T}^2 \times [0, 1]$ . The foliation  $\mathcal{F}_\theta$  will be the foliation obtained by gluing  $\mathbb{T}^2 \times \{0\}$  with  $\mathbb{T}^2 \times \{1\}$  by the diffeomorphism

$$\psi_\theta : \mathbb{T}^2 \times \{0\} \rightarrow \mathbb{T}^2 \times \{1\} \quad \psi_\theta(t, s, 0) = (t + \theta, s + kt, 1)$$

**Remark.** Notice that if  $\mathcal{W}$  is a foliation of  $M_\psi$  which is transverse to  $T$  the torus obtained by projection of  $\mathbb{T}^2 \times \{0\} \sim \mathbb{T}^2 \times \{1\}$  we know that it must be invariant by a map of  $T$  which is isotopic to  $\psi$ .

The foliation  $\mathcal{F}_\infty$  is the foliation by the fibers of the torus bundle. These foliations correspond to the foliations of the form  $\mathcal{F}_\pi$  in  $\mathcal{H}/\Gamma$  defined in section 5. In particular, we know that they are pairwise not almost parallel.

**Theorem B.5.** *Let  $\mathcal{W}$  be a codimension one Reebless foliation of  $M_\psi$  where  $\psi$  is a Dehn twist as above. Then,  $\mathcal{W}$  is almost aligned with  $\mathcal{F}_\theta$  for some  $\theta \in \mathbb{R} \cup \{\infty\}$ . Moreover, if  $\theta$  is irrational then  $\mathcal{W}$  is almost parallel to  $\mathcal{F}_\theta$ .*

*Proof.* If  $\mathcal{W}$  has a torus leaf, this torus must be incompressible by Novikov's Theorem ([So, CC]). We can cut the foliation along this torus. By doing the same doubling procedure as in Proposition B.4 we obtain a foliation of  $\mathbb{T}^3$  and using Theorem B.3 we deduce that  $\mathcal{W}$  is almost aligned with a foliation of the form  $\mathcal{F}_\theta$  with  $\theta$  being irrational.

If  $\mathcal{W}$  has no torus leaves, we can consider the torus  $\mathbb{T}^2 \times \{0\} \subset M_\psi$  which is incompressible. Using Theorem B.2 we can make an isotopy and assume that the foliation  $\mathcal{W}$  is transverse to this torus (recall from Proposition B.1 that the isotopy does not affect the equivalence class of the foliation under the relation of being almost parallel). Here we are using the fact that the isotopy of the torus can be extended to a global isotopy of  $M$  (see for example Theorem 8.1.3 of [Hi]).

We can cut  $M_\psi$  by this torus and apply Proposition B.4 to obtain that  $\mathcal{W}$  in  $\mathbb{T}^2 \times [0, 1]$  is almost aligned to a linear foliation of  $\mathbb{T}^2$  times  $[0, 1]$ . In fact, if the foliation is not almost parallel to the linear foliation we deduce that the foliation in  $\mathbb{T}^2 \times \{0\}$  must have

Reeb annuli (see section 4.A of [Pot<sub>2</sub>]). Since the foliation in  $\mathbb{T}^2 \times \{0\}$  must be glued by  $\psi$  with the foliation in  $\mathbb{T}^2 \times \{1\}$  we deduce that it must permute these Reeb annuli which are finitely many. So, we get that there is a periodic circle of the foliation of  $\mathbb{T}^2 \times \{0\}$  by  $\psi$  which implies the existence of a torus leaf for  $\mathcal{W}$ . We deduce that  $\mathcal{W}$  in  $\mathbb{T}^2 \times [0, 1]$  is almost parallel to a linear foliation of  $\mathbb{T}^2$  times  $[0, 1]$ .

Now, we must show that this linear foliation corresponds to the linear foliation  $\mathcal{L}$  by circles of the form  $\{t\} \times S^1$  but this follows from the fact that the foliation is invariant by  $\psi$ .

Now we must see that after gluing the foliation is almost parallel to some  $\mathcal{F}_\theta$ . This follows from the following fact, since in the boundary it is almost parallel to the foliation  $\mathcal{L}$ , we know that it has at least one circle leaf  $L$ . Now we obtain the value of  $\theta$  by regarding the relative order of the images of  $\psi^n(L)$  and performing a classical rotation number argument as in the circle.  $\square$

When  $\psi$  is isotopic to Anosov, the classification gives only three possibilities.

We consider then  $A$  a hyperbolic matrix in  $SL(2, \mathbb{Z})$  and in  $M_A$  we consider the following linear foliations:  $\mathcal{F}_A^{cs}$  is given by the linear foliation which is the projection of  $\mathcal{L}^s \times [0, 1]$  where  $\mathcal{L}^s$  is the linear foliation corresponding to the strong stable foliation of  $A$  and similarly we obtain  $\mathcal{F}_A^{cu}$  as the projection of  $\mathcal{L}^u \times [0, 1]$  where  $\mathcal{L}^u$  is the linear foliation which corresponds to the strong unstable foliation.

Finally, we consider the foliation  $\mathcal{F}_T$  which is the projection of foliation by tori  $\mathbb{T}^2 \times \{t\}$  to  $M_A$ . Clearly, these 3 foliations are pairwise not almost parallel to each other.

**Theorem B.6.** *Let  $\mathcal{W}$  be a Reebless foliation of  $M_A$ , then,  $\mathcal{W}$  is almost aligned to one of the foliations  $\mathcal{F}_A^{cs}, \mathcal{F}_A^{cu}$  or  $\mathcal{F}_T$ . Moreover, if  $\mathcal{W}$  has no torus leaves, then  $\mathcal{W}$  is almost parallel to either  $\mathcal{F}_A^{cs}, \mathcal{F}_A^{cu}$  and if it is almost aligned with  $\mathcal{F}_T$  it has a torus leaf.*

*Proof.* The first part of the proof is as in the previous Theorem: If  $\mathcal{W}$  has a torus leaf, it must be isotopic to  $T$  the projection of  $\mathbb{T}^2 \times \{0\}$  since it is the only incompressible embedded torus in  $M_A$  and we get that  $\mathcal{W}$  is almost parallel to  $\mathcal{F}_T$ .

Otherwise, we can assume that  $\mathcal{W}$  is transverse to  $T$  and we obtain a foliation of  $\mathbb{T}^2 \times [0, 1]$  which is almost aligned with a foliation of the form  $\mathcal{L} \times [0, 1]$  and which in  $T$  is invariant under a diffeomorphism  $f$  isotopic to  $A$ .

This implies that the linear foliation  $\mathcal{L}$  is either the strong stable or the strong unstable foliation for  $A$ , and in particular, since it has no circle leaves, we get that  $\mathcal{W}$  in  $\mathbb{T}^2 \times [0, 1]$  is almost parallel to  $\mathcal{L} \times [0, 1]$ .

Now, since the gluing map  $f$  is isotopic to  $A$ , we know it is semiconjugated to it, so, we get that after gluing, the foliations remain almost parallel.  $\square$

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